

Diffeomorphism Invariant Integrable Field Theories and Hypersurface Motions in Riemannian Manifolds

Martin Bordemann

Fakultät für Physik
Universität Freiburg
Hermann-Herder-Str. 3
79104 Freiburg i. Br., F. R. G
e-mail: mbor@phyq1.physik.uni-freiburg.de

Jens Hoppe¹

Institut für Theoretische Physik
ETH Hönggerberg
CH 8093 Zürich, Switzerland
e-mail: hoppe@itp.phys.ethz.ch

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Abstract

We discuss hypersurface motions in Riemannian manifolds whose normal velocity is a function of the induced hypersurface volume element and derive a second order partial differential equation for the corresponding time function $\tau(x)$ at which the hypersurface passes the point x . Equivalently, these motions may be described in a Hamiltonian formulation as the singlet sector of certain diffeomorphism invariant field theories. At least in some (infinite class of) cases, which could be viewed as a large-volume limit of Euclidean M -branes moving in an arbitrary $M + 1$ -dimensional Riemannian manifold, the models are integrable: In the time-function formulation the equation becomes linear (with $\tau(x)$ a harmonic function on the embedding Riemannian manifold). We explicitly compute solutions to the large volume limit of Euclidean membrane dynamics in \mathbb{R}^3 by methods used in electrostatics and point an additional gradient flow structure in \mathbb{R}^n . In the Hamiltonian formulation we discover infinitely many hierarchies of integrable, multidimensional, N -component theories possessing infinitely many diffeomorphism invariant, Poisson commuting, conserved charges.

¹Heisenberg Fellow

On leave of absence from Karlsruhe University

1. In physics, and mathematics, surface motions are usually considered independent of the parametrisation, the most prominent example perhaps being the ‘flow by mean curvature’ which has been of intense mathematical interest [1] as well as of physical importance [2]. When studying the dynamics of relativistic membranes, on the other hand, one encounters [4] in a partially gauge-fixed formulation, a very interesting equation (describing the time-evolution of a 2-dimensional surface in \mathbb{R}^3) which is parametrisation-*dependent* (meaning that different parametrisations of the initial surface Σ_0 generically lead to geometrically different shapes Σ_t at later times, t (in the case at hand there is a residual symmetry group of area-preserving diffeomorphisms, but any reparametrisation whose Jacobi determinant is different from 1 will generically change the motion); alternatively, the equation (though first order in time) may be viewed as requiring the initial unparametrized shape Σ_0 *and* the initial unparametrized normal velocity field on Σ_0 as initial conditions (the parametrisation of Σ_0 , and of Σ_t , then follows, up to area-preserving diffeomorphisms of Σ_0 -that don’t change the geometry of the motion- from the equation). Moreover, it was shown in [7] that this seemingly parametrisation dependent equation can be obtained from the singlet sector of a diffeomorphism invariant Hamiltonian field theory. In [6], on the other hand, some surface motions in \mathbb{R}^3 were shown to be best described by a (second order partial differential) equation for $\tau(\vec{x})$, the time at which Σ_t contains \vec{x} .

In this letter, we significantly generalize this situation (to higher dimensions, curved embedding spaces, and more general dynamics) by considering a rather general class of parametrisation dependent hypersurface motions in Riemannian manifolds. For these hypersurface motions we shall give an equivalent diffeomorphism-invariant (constrained) Hamiltonian formulation as well as derive a second order partial differential equation for the time-function $\tau(x)$, which in some cases (that may be viewed as large-volume limits of Euclidean M -branes or, equivalently, as highly generalized, multilinear, field theoretic extensions of Nahm’s matrix equations [8]) turns out to simply be $\Delta\tau = 0$ where Δ is the Laplacean of the embedding Riemannian manifold in which the motion takes place. In the particular case of three-dimensional Euclidean space the solution of the Laplace equation for the time function becomes easy to handle thanks to the formal equivalence with the potential equation of electrostatics: we compute explicit solutions of these surface motions determined by singular one-dimensional membranes at $t = -\infty$ and $t = +\infty$ (corresponding to negatively and positively charged composed loops of wire, respectively) between which two-dimensional surfaces develop in time.

Finally, and perhaps most importantly, we are led to an infinite set of hierarchies of multidimensional integrable diffeomorphism invariant N -component field theories for which we are able to provide a complete set of Poisson-commuting, diffeomorphism invariant, conserved charges.

2. Let Σ be an orientable compact connected manifold of dimension M with

a fixed volume form ρ . We shall denote co-ordinates on Σ by $\varphi^r, 1 \leq r \leq M$. Let (\mathcal{N}, ζ) be an orientable Riemannian manifold of dimension $N = M + 1$ on which we shall denote co-ordinates by $x^i, 1 \leq i \leq M + 1$ and whose volume form induced by ζ will be denoted by ω_ζ . Consider the set of smooth immersions $x : \Sigma \rightarrow \mathcal{N}, \text{imm}(\Sigma, \mathcal{N})$. For each immersion x there is a unique outward normal field $n[x] : \Sigma \rightarrow T\mathcal{N}$ (depending on x and its first derivatives) attached to the parametrized hypersurface $x : \Sigma \rightarrow \mathcal{N}$. Moreover, let $x^*\zeta$ denote the Riemannian metric on Σ obtained by pulling back ζ to Σ by x and denote by $\omega_{x^*\zeta}$ the corresponding volume form on Σ . It follows that there is a unique smooth positive function $\sqrt{g[x]}/\rho : \Sigma \rightarrow \mathbb{R}$ depending on x , its first derivatives, and the reference volume form ρ such that $\omega_{x^*\zeta} = (\sqrt{g[x]}/\rho)\rho$. Finally, let α be a positive smooth function on (a suitable open interval of) the positive real line with nowhere vanishing derivative. Consider now the hypersurface motion $(-\epsilon, \epsilon) \rightarrow \text{imm}(\Sigma, \mathcal{N}) : t \mapsto x(t)$ in \mathcal{N} described by the following differential equation:

$$\dot{x} := \frac{\partial x}{\partial t} = \alpha(\sqrt{g[x]}/\rho) n[x] \quad . \quad (1)$$

In local co-ordinates this equation looks as follows: writing

$$g_{rs} = \frac{\partial x^i}{\partial \varphi^r} \frac{\partial x^j}{\partial \varphi^s} \zeta_{ij}(x) \quad (2)$$

for the induced metric $x^*\zeta$ we get

$$\begin{aligned} \dot{x}^i &= \alpha(\sqrt{\det(g_{rs})}/\rho) \\ &\quad \frac{1}{M! \sqrt{\det(g_{rs})}} \zeta^{ij}(x) \sqrt{\det(\zeta_{kl})}(x) \epsilon_{ji_1 \dots i_M} \epsilon^{r_1 \dots r_M} \frac{\partial x^{i_1}}{\partial \varphi^{r_1}} \dots \frac{\partial x^{i_M}}{\partial \varphi^{r_M}} \end{aligned} \quad (3)$$

The following choices of α are of particular interest:

$$\text{Lorentzian } M\text{-brane} \quad : \quad \dot{x} = \sqrt{1 - (\sqrt{g[x]}/\rho)^2} n[x] \quad (4)$$

$$\text{Euclidean } M\text{-brane} \quad : \quad \dot{x} = \sqrt{(\sqrt{g[x]}/\rho)^2 - 1} n[x] \quad (5)$$

and the large-volume limit of (5),

$$\dot{x} = (\sqrt{g[x]}/\rho) n[x]. \quad (6)$$

The small-volume limit of (4),

$$\dot{x} = n[x] \quad (7)$$

(which may be called the *Optical model*) is not contained in this class (since the prefactor α of the normal is constant). Nevertheless, it can be solved directly since (7) is easily seen to imply the following equation of free motion:

$$\frac{\partial^2 x^i}{\partial t^2} + \Gamma_{jk}^i(x) \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial t} = 0 \quad (8)$$

(where Γ_{jk}^i denote the Christoffel symbols of ζ) which can be solved as soon as the geodesic flow of (\mathcal{N}, ζ) is explicitly known (e.g. for flat \mathbb{R}^N , the N -sphere, etc.). Some of the following results will also remain true for (7).

As in [6] we are trying to derive a differential equation for the time-function:

Theorem 0.1 *Let Σ , ρ , \mathcal{N} , ζ and α be defined as above.*

1. *Suppose that there is a positive real number ϵ and a solution x of the above equation (1) such that the map $x : (-\epsilon, \epsilon) \times \Sigma \rightarrow \mathcal{N} : (t, \varphi) \mapsto x(t, \varphi)$ is a diffeomorphism onto its image, \mathcal{N}_ϵ , which is an open neighbourhood of the initial hypersurface $x(0, \Sigma)$.*

Then the first component of the inverse map $x^{-1} : \mathcal{N}_\epsilon \rightarrow (-\epsilon, \epsilon) \times \Sigma$ which we shall call the time function τ satisfies the following equations where $\nabla\tau := \zeta^\#(d\tau)$ ($\nabla^i\tau = \zeta^{ij}\partial_j\tau$) denotes the gradient of τ , $|v| := \sqrt{\zeta(v, v)} = \sqrt{\zeta_{ij}v^iv^j}$ for any tangent vector v to \mathcal{N} , and the symbol “ \cdot ” stands for covariant derivative.

$$n[x] = \frac{\nabla\tau}{|\nabla\tau|}(x) \quad (9)$$

$$|\nabla\tau|(x) = \frac{1}{\alpha(\sqrt{g[x]}/\rho)}, \quad (10)$$

$$0 = |\nabla\tau|^2\zeta^{ij}\tau_{;ij} + (\tilde{\beta}(|\nabla\tau|) - 1)\tau_{;ij}\nabla\tau^i\nabla\tau^j. \quad (11)$$

where $\tilde{\beta}$ is defined by:

$$\tilde{\beta}(z) := -z \frac{\partial}{\partial z}(\alpha^{-1}(1/z)). \quad (12)$$

Eqs (9) and (10) remain true for positive α whose derivative may vanish.

2. *Conversely, suppose that the smooth function $\tau : \mathcal{N} \rightarrow \mathbb{R}$ is a solution of the second order partial differential equation (11) and that there is a positive real number ϵ such that all its level surfaces $\tau = c$ for $c \in (-\epsilon, \epsilon)$ are diffeomorphic to Σ .*

Then there is a parametrisation of the zero level surface Σ_0 of τ , $x_0 : \Sigma \rightarrow \Sigma_0$ and a positive real number r such that eqn (10) is satisfied for x_0 and $r\rho$. Moreover, let Φ_t denote the flow of the vector field $X := \nabla\tau/|\nabla\tau|^2$, i.e. the solution of the ordinary differential equation $\partial\Phi_t(x)/\partial t = X(\Phi_t(x))$ with initial condition $\Phi_0(x) = x$. Then for each parametrisation x_0 of the zero level surface which satisfies the above condition (10) the map $x : (-\epsilon, \epsilon) \times \Sigma \rightarrow \mathcal{N}$ defined by

$$x(t, \varphi) := \Phi_t(x_0(\varphi)) \quad (13)$$

satisfies equation (1) (with ρ replaced by $r\rho$) with initial condition x_0 .

Proof: 1. It is obvious that the gradient of τ will be orthogonal on the surfaces of constant time whence the gradient of τ is proportional to the surface normal $n[x]$ which gives (9). Using the inverse function theorem

$$\delta_j^i = \frac{\partial x^i}{\partial t} \frac{\partial \tau}{\partial x^j} + \frac{\partial x^i}{\partial \varphi^r} \frac{\partial \varphi^r}{\partial x^j},$$

contracting with $\zeta_{ik} n[x]^k$, and using equation (1) and the definition of the surface normal we get the second equation (10). This leads to

$$\dot{x} = \frac{\nabla \tau}{|\nabla \tau|^2}(x) \quad (14)$$

In order to obtain a differential equation for the time function τ in the case where the derivative of the function α does never vanish we differentiate equation (10) with respect to time and use equation (14) to get rid of \dot{x} . Thereby the time derivative of the left hand side of (10) can solely be expressed by second covariant derivatives of the time function. To do the same for the right hand side we use the equation

$$\sqrt{\dot{g}[x]} = \frac{1}{2} \sqrt{g[x]} g^{rs} \dot{g}_{rs},$$

obtain by the chain rule

$$\partial_r \dot{x}^i = \partial_j (\nabla \tau^i / |\nabla \tau|^2) \partial_r x^j$$

observe that the orthogonal projection π onto the tangent spaces of the hyper-surfaces can be expressed in two ways by

$$g^{rs} \partial_r x^i \partial_s x^j = \pi^{ij}(x) = \zeta^{ij}(x) - n[x]^i n[x]^j$$

(see e.g. [4]), and replace the normal vectors by the normalized gradients of the time-function (9) which gives the second order equation (11).

2. Denote by Σ_0 the level surface $\tau = 0$ and by $i : \Sigma_0 \rightarrow \mathcal{N}$ the canonical inclusion. Let $\omega_{i^* \zeta}$ denote the volume form on Σ_0 induced by the metric ζ . Consider the modified volume form

$$\frac{1}{\alpha^{-1}(\frac{1}{|\nabla \tau|(i)})} \omega_{i^* \zeta}$$

on Σ_0 . Let r be the unique positive real number such that the two following integrals are equal:

$$\int_{\Sigma} r \rho = \int_{\Sigma_0} \frac{1}{\alpha^{-1}(\frac{1}{|\nabla \tau|(i)})} \omega_{i^* \zeta}$$

By Moser's lemma (see [11]) there is a diffeomorphism $x_0 : \Sigma \rightarrow \Sigma_0$ such that the two volume forms are diffeomorphic, i.e.:

$$r \rho = x_0^* \left(\frac{1}{\alpha^{-1}(\frac{1}{|\nabla \tau|(i)})} \omega_{i^* \zeta} \right) = \frac{1}{\alpha^{-1}(\frac{1}{|\nabla \tau|(x_0)})} \omega_{x_0^* \zeta}$$

whence eqn (10) is satisfied for $t = 0$ and $r \rho$.

Observe now that for all points $y \in \Sigma_0$ we have

$$\tau(\Phi_t(y)) = t$$

which can be seen by differentiating the left hand side with respect to t and showing it to be equal to 1. When we insert $y = x_0(\varphi)$ in this equation and differentiate with respect to φ^r we see that the gradient of the time function is always orthogonal to the surfaces $\Phi_t(x_0(\Sigma))$. Hence $\nabla\tau(x(t, \varphi))/|\nabla\tau(x(t, \varphi))|$ is equal to the hypersurface normal $n[x_t](\varphi)$ whence we have the differential equation

$$\dot{x}(t, \varphi) = \frac{\nabla\tau(x(t, \varphi))}{|\nabla\tau(x(t, \varphi))|^2} = \frac{1}{|\nabla\tau(x(t, \varphi))|} n[x_t](\varphi)$$

We have already shown above that the function (for fixed $\varphi \in \Sigma$)

$$f(t) := \alpha(\sqrt{g_{rs}(t, \varphi)}/(r\rho(\varphi))) |\nabla(x(t, \varphi))|$$

equals 1 for $t = 0$ by construction of x_0 . Upon differentiating f with respect to t , eliminating \dot{x} by the equation above and using the second order equation (11) we get the first order time dependent differential equation for f :

$$\dot{f}(t) = \frac{\tau_{;ij} \nabla\tau^i \nabla\tau^j \tilde{\beta}(|\nabla\tau|)}{|\nabla\tau|^4} (x(t, \varphi)) \left(\frac{1}{\tilde{\beta}(|\nabla\tau|)(x(t, \varphi))} - \frac{1}{\tilde{\beta}(\frac{|\nabla\tau|(x(t, \varphi))}{f(t)})} \right) f(t)$$

Since the right hand side smoothly depends on f in a neighbourhood of $f = 1$ the obvious constant solution $f(t) = 1$ is unique. This proves eqn (10) for all t .

Q.E.D.

In other words, the open neighbourhood \mathcal{N}_ϵ will be foliated by the level sets of τ which are nothing else but the surfaces of constant time. Eqs (9) and (14) have also been considered in [9] and [10] and only use the fact that \dot{x} is proportional to the unit normal.

Note that for nonconstant α the above surface motion is *not* reparametrisation invariant, which is here reflected in the fact that the time function satisfies a second order partial differential equation whose normal derivative at the zero level surface encodes the “volume part” of the parametrisation, i.e. the induced volume form compared to a reference volume. On the other hand, for the Optical model (7) the function α is equal to one: this model of surface motion is invariant under reparametrisation, and the differential equation for the time-function is first order, viz. eqn (10) for $\alpha = 1$ which is the eikonal equation $|\nabla\tau|^2 = 1$ known in optics.

The second order equations for all the time-functions dealt with in Theorem 0.1 can be derived from a Lagrangean: define the real-valued smooth function F as any solution of the first order differential equation:

$$F'(z) = \frac{C}{\alpha^{-1}(\frac{1}{z})} \tag{15}$$

where C is an arbitrary nonzero real number. Define the Lagrangean \mathcal{L} for the time-function τ as:

$$\mathcal{L}(\partial\tau) := \sqrt{\det(\zeta_{ij})} F(|\nabla\tau|) \quad (16)$$

A straight forward calculation shows that the Euler Lagrange equations produce (11) for all F solving (15).

For the model (6) we can choose the Lagrangean $\mathcal{L}(\partial\tau)$ equal to $\sqrt{\det(\zeta_{ij})} \frac{1}{2} |\nabla\tau|^2$ and get the Laplace equation

$$\Delta\tau = 0 \quad (17)$$

where Δ is the Laplacean operator $\frac{1}{\sqrt{\det(\zeta_{ij})}} \partial_i (\sqrt{\det(\zeta_{ij})} \zeta^{ij} \partial_j)$. This means that -despite its complicated nonlinear appeal- the model (6) is integrable at least for all those Riemannian manifolds for which the Laplace equation can be solved in a sufficiently explicit way, e.g. for \mathbb{R}^n , the sphere S^n and hyperbolic space H^n . We shall discuss the surface motion in \mathbb{R}^3 (for which a linearizability was already noted in [5] without using the time function) according to its harmonic time-function in more detail in the next section.

In numerical relativity theory the foliation of a given spacetime into spacelike hypersurface which are level surfaces of a (Lorentz) harmonic time-function has been dealt with by C. Bona and J. Massó in [3]. This method of ‘harmonic slicing’ avoids the occurrence of spacetime singularities on these surfaces.

For the Lorentzian membrane we can choose the Lagrangean equal to the function $\sqrt{\det(\zeta_{ij})} \sqrt{|\nabla\tau|^2 - 1}$ and get the Euler-Lagrange equations

$$(1 - |\nabla\tau|^2) \Delta\tau + \tau_{;ij} \nabla\tau^i \nabla\tau^j = 0 \quad (18)$$

whereas for the Euclidean membrane we can choose $\mathcal{L}(\partial\tau)$ equal to the function $\sqrt{\det(\zeta_{ij})} \sqrt{1 + |\nabla\tau|^2}$ and get the equation

$$(1 + |\nabla\tau|^2) \Delta\tau - \tau_{;ij} \nabla\tau^i \nabla\tau^j = 0. \quad (19)$$

Note that the global description of surface motion by parametrisation may drastically differ from the description by the level sets of a time function: although being equivalent for short time intervals (and some technical assumptions, see Thm 0.1) the former allows the membrane to be at the same position at different times, but fixes the topological type, whereas the latter allows for varying topological type, but every point in space is contained in at most one level surface. In both pictures one has to admit violations of the regularity of the mappings (i.e. points where the Jacobians cease to have maximal rank) in order to produce interesting situations.

3. In this section we should like to discuss some solutions of the model (6) in \mathbb{R}^3 in terms of its harmonic time-function. A comparison with the Euclidean membrane model (5) shows that (6) is an approximation of the Euclidean membrane model in the regime where the induced spatial membrane area element $\sqrt{\det(g_{rs})}$ is large compared to a reference area element ρ .

The Laplace equation (17) for flat \mathbb{R}^3 can be treated by methods known in electrostatics (see e.g. [12]). For example, consider a piece of wire of length L which is uniformly charged with line charge $\lambda = q/L$ and which is lying along the z -axis between $-p := L/2$ and p : its electrostatic potential τ is given by

$$\tau(\vec{x}) = \lambda \log \left(\frac{\sqrt{x^2 + y^2 + (z+p)^2} + z + p}{\sqrt{x^2 + y^2 + (z-p)^2} + z - p} \right) \quad (20)$$

This function is harmonic outside the interval $[-p, p]$ along the z -axis on which it becomes $\pm\infty$ according to the sign of λ and tends to zero for large distances from the origin. It is known that all the equipotential surfaces of this function are axially symmetric ellipsoids whose focal points are the end-points of the piece of wire. In case λ is negative we get the -at least in electrostatics- unusual picture of the equipotential surfaces as surfaces of constant time: for $\tau = -\infty$ it is a singular line which blows up into bigger and bigger ellipsoids, and at $\tau = 0$ the level surface is the sphere at spatial infinity.

This situation can be generalized a bit: let

$$\vec{x}_i^-, \vec{x}_j^+ : S^1 \rightarrow \mathbb{R}^3, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n \quad (21)$$

be closed curves in \mathbb{R}^3 which are either points or immersed with only a finite number of self-intersection points. Suppose furthermore that any two different curves do not intersect. Moreover, let $\lambda_i^-, \lambda_j^+ : S^1 \rightarrow \mathbb{R}$ be positive smooth functions where $1 \leq i \leq m, \quad 1 \leq j \leq n$. On \mathbb{R}^3 minus all the images of the closed curves, $\vec{x}_1^-(S^1), \dots, \vec{x}_m^-(S^1), \vec{x}_1^+(S^1), \dots, \vec{x}_n^+(S^1)$ we can define the following harmonic function τ :

$$\tau(\vec{x}) := - \sum_{i=1}^m \int_0^{2\pi} d\varphi \frac{\lambda_i^-(\varphi)}{|\vec{x} - \vec{x}_i^-(\varphi)|} + \sum_{j=1}^n \int_0^{2\pi} d\varphi \frac{\lambda_j^+(\varphi)}{|\vec{x} - \vec{x}_j^+(\varphi)|} \quad (22)$$

(It is harmonic because $\vec{x} \mapsto \frac{1}{|\vec{x} - \vec{y}|}$ is harmonic for all $\vec{x} \neq \vec{y}$.) From the point of view of electrostatics this a finite collection of closed curves carrying either positive or negative line charges. Due to the presence of the denominators in the above formula for τ it is clear that there is a negative real number $-t_0$ of very large absolute value such that the equipotential surface $\tau = -t_0$ breaks up into m connected surfaces located each in the vicinity of one of the “incoming curves” $\vec{x}_i^-(S^1)$. The following intuitive argument shows that each connected piece is diffeomorphic to a Riemann surface of genus equal to the number of loops which are generated by the self-intersections. In other words: the level surface piece near such a curve looks like a two-sphere if the curve is just a point, like a two-torus if the curve is immersed with no self-intersection, like a surface of genus two if the curve looks like the figure 8, and so forth. Indeed, since $\vec{x}_i^-(S^1)$ is compact we can cover it by a finite number of balls of diameter L where L can be chosen arbitrarily small. Now suppose that the map \vec{x}_i^- is well-behaved

enough so that we can approximate it by a sequence of straight pieces of wire each carrying a constant line charge equal to the mean value of λ_i^- over the piece of the curve. Formula (20) gives an approximate solution of the potential τ . From this formula one can infer that if the distance of the point \vec{x} to the curve is very small compared to L the potential is approximately proportional to the natural logarithm of the distance. Hence for “very early” values of τ the connected components of the equipotential surface are approximated by surfaces of constant distance from a given curve which visibly have the required topology. One has the same argumentation for very large or “very late” values of τ : here the connected components of the equipotential surfaces are certain Riemann surfaces centered around the “outgoing curves” $x_j^+(S^1)$.

We can interpret this geometrical picture as a theory for propagating membranes with prescribed change of topology: if the time equals $-\infty$ one has m incoming membranes shrunk to one-dimensional objects whose homotopy and initial “shape” are encoded in the curves \vec{x}_i^- with line charge functions λ_i^- . These m membranes will grow bigger and possibly melt together (which can be interpreted as interaction) and for large positive time values the surface of equal time decomposes again into n connected components which will eventually center around the outgoing curves \vec{x}_j^+ with final “shape” described by λ_j^+ and will be equal to these curves for $t = +\infty$.

In order to get more general solutions one has to solve the Dirichlet problem for the potential τ outside m conducting Riemann surfaces held at value $-t_0$ and n conducting Riemann surfaces held at value $+t_0$ for a positive real number t_0 whose solution in principle exists and is unique.

In order to get explicit solutions for the model (6) in higher dimensional Euclidean space \mathbb{R}^{M+1} we can easily generalize the above construction (22): we have to replace the curves $\vec{x}_i^\mp(\varphi)$ and by (at most) $M-1$ dimensional surfaces $\vec{x}_i^\mp(\varphi^1, \dots, \varphi^{M-1})$ equipped with higherdimensional “line charge” functions and the line integral over $1/|\vec{x} - \vec{x}_i^\mp(\varphi)|$ by a surface integral (of codimension two) over $1/|\vec{x} - \vec{x}_i^\mp(\varphi^1, \dots, \varphi^{M-1})|^{M-1}$. More generally, one has to explicitly solve the higher dimensional analogue of the above-mentioned Dirichlet problem whose solution in principle exists and is unique.

Finally one should point out that (6) may also be written as

$$\dot{x}^i = \int_{\Sigma} d^M \tilde{\varphi} H^{ij}(\varphi, \tilde{\varphi}) \frac{\delta W}{\delta x^i(\tilde{\varphi})}[x] \quad (23)$$

with

$$H^{ij}(\varphi, \tilde{\varphi}) := \frac{\delta^{ij} \delta^M(\varphi, \tilde{\varphi})}{\rho(\varphi)} \quad (24)$$

$$W[x] := \frac{1}{M+1} \int_{\Sigma} d^M \varphi x^i(\varphi) \epsilon_{ii_1 \dots i_M} \epsilon^{r_1 \dots r_M} \frac{1}{M!} \partial_{r_1} x^{i_1}(\varphi) \dots \partial_{r_M} x^{i_M}(\varphi) \quad (25)$$

in \mathbb{R}^n , i.e. are gradient flows with respect to the volume functional W and the metric $H^{ij}(\varphi, \tilde{\varphi})$.

4. Consider now a Hamiltonian field theory with fields (x, p) from Σ into the cotangent bundle $T^*\mathcal{N}$ of \mathcal{N} where the conjugate momentum field p is supposed to be densitized.² Let the Hamiltonian be of the form

$$H(x, p) := \int_{\Sigma} d^M \varphi \sqrt{g}[x] h\left(\frac{|p|}{\sqrt{g}[x]}\right) \quad (26)$$

where h is a smooth function of one variable, $\sqrt{g} := \sqrt{g}[x] := \sqrt{\det g_{rs}}$ is defined as in (2), $p^2 := p_i p_j \zeta^{ij}(x)$, $|p| := \sqrt{p^2}$, and u will be short for $\frac{|p|}{\sqrt{g}}$.³

From the canonical Poisson structure

$$\{F, G\} = \int_{\Sigma} d^M \varphi \left(\frac{\delta F}{\delta x^i(\varphi)} \frac{\delta G}{\delta p_i(\varphi)} - \frac{\delta G}{\delta x^i(\varphi)} \frac{\delta F}{\delta p_i(\varphi)} \right) \quad (27)$$

we obtain Hamiltonian equations of motion

$$\frac{\partial x^i}{\partial t} = \zeta^{ij}(x) h' \frac{p_j}{|p|} \quad (28)$$

$$\begin{aligned} \frac{\partial p_i}{\partial t} = & \zeta_{ij}(x) \partial_r ((h - uh') \sqrt{g} g^{rs} \partial_s x^j) \\ & + \zeta_{ij}(x) (h - uh') \sqrt{g} g^{rs} \partial_r x^k \partial_s x^l \Gamma_{kl}^j(x) \\ & - \frac{1}{2} \partial_i \zeta^{jk}(x) h' \frac{p_j p_k}{|p|} \end{aligned} \quad (29)$$

Defining

$$C_r(x, p) := p_i \partial_r x^i \quad (30)$$

(the infinitesimal generators of diffeomorphisms) it is easy to check that (28) and (29) imply

$$\frac{\partial C_r}{\partial t} = 0 \quad (31)$$

(H is invariant under diffeomorphisms $\phi : \Sigma \rightarrow \Sigma$), as well as

$$\frac{\partial}{\partial t} (\sqrt{g} h) = \partial_r \left(\left(\frac{h h'}{u} - h'^2 \right) g^{rs} C_s(x, p) \right). \quad (32)$$

²Although we shall be always working in canonical x^i and p_i co-ordinates (a bundle chart) on $T^*\mathcal{N}$ this has a global meaning: the pair (x, p) is a smooth vector bundle homomorphism $\wedge^M T\Sigma \rightarrow T^*\mathcal{N}$ over the map $x : \Sigma \rightarrow \mathcal{N}$, i.e. $x \circ \wedge^M \tau_{\Sigma} = \tau_{\mathcal{N}}^* \circ (x, p)$ with the obvious bundle projections.

³In order to avoid clumsy notation we shall henceforth suppress the arguments $\varphi \in \Sigma$ in the integrals, the field x and its derivatives in $\sqrt{g}[x]$ and $n[x]$, and the argument u of h and h' from time to time.

For solutions $(x(t), p(t))$ of the equations of motion satisfying $C_r(x(0), p(0)) = 0$ the r.h.s. of eqn (32) vanishes identically (for all t due to (31)), hence (for such solutions)

$$\sqrt{g}[x(t)]h\left(\frac{|p|(t)}{\sqrt{g}[x(t)]}\right) =: \rho \quad (33)$$

with ρ some *time independent* density on Σ . Inverting (33) to obtain $\frac{|p|}{\sqrt{g}}$ as a function of $\frac{\sqrt{g}}{\rho}$, and noting that for $N - M = 1$ the condition

$$C_r(x, p) = p_i \partial_r x^i = 0, \quad 1 \leq r \leq M \quad (34)$$

for x an immersion means that $\zeta^{ij} p_j$ must lie in the direction normal to the hypersurface defined by $\varphi \mapsto x(t, \varphi)$, one sees that (28) can be rewritten in the form mentioned in section 2 (see (1))

$$\frac{\partial x}{\partial t} = \alpha\left(\frac{\sqrt{g}}{\rho}\right)n \quad (35)$$

with the functions α and h being related via

$$\alpha(z)z^2\partial_z(h^{-1}(\frac{1}{z})) = -1. \quad (36)$$

In particular, for $h(z) = \sqrt{2z}$ we get $\alpha(z) = z$ (6) and

$$H[x, p] = \int_{\Sigma} d^M \varphi (4g[x]p^2)^{\frac{1}{4}}. \quad (37)$$

For $h(z) = \sqrt{1+z^2}$ which corresponds to $\alpha(z) = \sqrt{1-z^2}$ we get the Hamiltonian

$$H[x, p] = \int_{\Sigma} d^M \varphi \sqrt{p^2 + g[x]} \quad (38)$$

of the relativistic M -brane in $M + 1$ dimensional Minkowski space. $h(z) = z$ implying $\alpha(z) = 1$ gives the Hamiltonian

$$H[x, p] = \int_{\Sigma} d^M \varphi \sqrt{p^2} \quad (39)$$

of the Optical model (7).

Let us for a moment concentrate on (37): Motivated by the fact that the equation for the time function is linear (see 17) and the observation [6], made for $N = 3$, that a Lax pair formulation of $\partial \vec{x} / \partial t = \sqrt{g} \vec{n}$ implies the time-independance of $\int Q(x(t, \varphi^1, \varphi^2) d\varphi^1 d\varphi^2$ for harmonic polynomials $Q(x^1, x^2, x^3)$, one is led to the conjecture that for all $N = M + 1$, ρ , and $Q(x)$ any harmonic function of x ,

$$\mathbf{Q}[x] := \int_{\Sigma} d^M \varphi \rho Q(x) \quad (40)$$

will be independent of time if the hypersurface $\Sigma_t = x_t(\Sigma)$ evolves according to (6). The proof is a trivial application of Gauss' Theorem: assuming that for all t the moving surface Σ_t is a boundary of some open set V_t of \mathcal{N} and writing dS_t^i for the surface element we get

$$\begin{aligned}
\frac{\partial \mathbf{Q}[x(t)]}{\partial t} &= \int_{\Sigma} d^M \varphi \rho \partial_i Q(x(t)) \frac{\partial x^i}{\partial t} \\
&= \int_{\Sigma} d^M \varphi \sqrt{g}[x(t)] n^i[x(t)] \partial_i Q(x(t)) \\
&= \int_{\Sigma_t} dS_t^i \partial_i Q(x(t)) \\
&= \int_{V_t} d^N x \sqrt{\det(\zeta_{kl})} \partial_j (\zeta^{ji} \partial_i Q) \\
&= \int_{V_t} d^N x \sqrt{\det(\zeta_{kl})} \Delta Q = 0.
\end{aligned}$$

In the Hamiltonian formulation (37) the nondynamical density $\rho(\varphi)$ is simply replaced by the Hamiltonian density

$$\mathcal{H}[x, p](\varphi) := \sqrt{g}[x](\varphi) h\left(\frac{|p|(\varphi)}{\sqrt{g}[x](\varphi)}\right), \quad (41)$$

i.e. $\mathcal{H}[x, p] = (g[x]p^2)^{\frac{1}{4}}$ in the case of (37). Indeed, for any harmonic Q the following functionals on phase space,

$$\mathbf{Q}[x, p] := \int_{\Sigma} d^M \varphi (gp^2)^{\frac{1}{4}} Q(x), \quad (42)$$

will be time-independent for solutions $(x(t), p(t))$ satisfying (28), (29), and the constraint (34).

Do the quantities (42) Poisson-commute (on the reduced space)? The answer is 'yes' (in fact they form an infinite, presumably complete set of Poisson-commuting, reparametrisation invariant charges, thus providing the solution of a non-trivial diffeomorphism invariant field theory)—but let us first make some observations valid for all theories of the general form (26), i.e. arbitrary h , N , M , and ζ : With the canonical Poisson structure (27) one has the following Poisson brackets where f, \tilde{f} are arbitrary smooth real-valued functions on Σ and X, Y are arbitrary vector fields on Σ :

$$\begin{aligned}
\left\{ \int_{\Sigma} d^M \varphi X^r C_r, \int_{\Sigma} d^M \varphi Y^s C_s \right\} &= \int_{\Sigma} d^M \varphi [X, Y]^r C_r \\
\left\{ \int_{\Sigma} d^M \varphi f \mathcal{H}, \int_{\Sigma} d^M \varphi \tilde{f} \mathcal{H} \right\} &= \int_{\Sigma} d^M \varphi \left(\frac{h h'}{u} - h'^2 \right) \\
&\quad g^{rs} (f \partial_s \tilde{f} - \tilde{f} \partial_s f) \mathcal{H}
\end{aligned} \quad (43)$$

(44)

$$\left\{ \int_{\Sigma} d^M \varphi f \mathcal{H}, \int_{\Sigma} d^M \varphi Y^s C_s \right\} = - \int_{\Sigma} d^M \varphi Y^s \partial_s f \mathcal{H} \quad (45)$$

where $[X, Y]^r$ denotes the Lie bracket $\partial_s Y^r X^s - \partial_s X^r Y^s$ of X and Y .

While (43), together with (31), imply that in principle the Hamiltonian theory (26) may be reduced to a Hamiltonian theory on a diffeomorphism-invariant phase space (involving only $N - M$ fields and $N - M$ momenta) one should note that on the reduced phase space the density \mathcal{H} (41) can no longer be defined, unless integrated over particular functions. This explains that although $\dot{\mathcal{H}} = 0$ (32) for solutions of (28) and (29) satisfying the constraint (31), the Dirac bracket of \mathcal{H} with H (see e.g. [7] for the case (38)) would not give zero, due to (45). In particular, \mathcal{H} is *not* a conserved quantity in the Hamiltonian theory.

The charges \mathbf{Q} (42), however, *are* conserved on the reduced phase space: $\mathbf{Q}[x, p] = \int_{\Sigma} d^M \varphi \mathcal{H} Q(x)$ commutes with C_r ,

$$\begin{aligned} \left\{ \int_{\Sigma} d^M \varphi \mathcal{H} Q(x), \int_{\Sigma} d^M \varphi Y^r C_r \right\} &= - \int_{\Sigma} d^M \varphi Y^r \partial_r Q(x) \mathcal{H} \\ &\quad + \int_{\Sigma} d^M \varphi \mathcal{H} \partial_i Q(x) Y^r \partial_r x^i = 0 \end{aligned} \quad (46)$$

(irrespective of Q , h , and $N - M$; just use (45) with $\tilde{f} = Q(x)$ for the first term on the r.h.s of (46)), as well as with H (provided $C_r = 0$, $N = M + 1$, and $hh' = \text{const}$). Moreover,

$$\begin{aligned} &\left\{ \int_{\Sigma} d^M \varphi \mathcal{H} Q(x), \int_{\Sigma} d^M \varphi \mathcal{H} \tilde{Q}(x) \right\} \\ &= \int_{\Sigma} d^M \varphi \left(\frac{hh'}{u} - h'^2 \right) g^{rs} (Q(x) \partial_s \tilde{Q}(x) - \tilde{Q}(x) \partial_s Q(x)) C_r \\ &\quad - \int_{\Sigma} d^M \varphi \sqrt{g}[x] h' h \frac{p_i}{|p|} \zeta^{ij}(x) (Q(x) \partial_j \tilde{Q}(x) - \tilde{Q}(x) \partial_j Q(x)) \\ &= \text{const.} \int_{\Sigma} d^M \varphi \sqrt{g}[x] n^j[x] (Q(x) \partial_j \tilde{Q}(x) - \tilde{Q}(x) \partial_j Q(x)) \\ &= \text{const.} \int_{V_t} d^N x \sqrt{\zeta_{kl}} \partial_i (\zeta^{ij} (Q \partial_j \tilde{Q} - \tilde{Q} \partial_j Q)) = 0 \end{aligned} \quad (47)$$

where again we have assumed that Σ_t is the boundary of V_t . In (47), the first equality is general (using (44), with $f = Q(x)$, $\tilde{f} = \tilde{Q}(x)$, for the first term), the second equality holds if $C_r = 0$ ($1 \leq r \leq M$) and $N = M + 1$ (so that $\zeta^{ij} p_j$ is proportional to the surface normal), and the last equality holds if Q and \tilde{Q} are harmonic functions (it actually holds as long as $\tilde{Q} \Delta Q - Q \Delta \tilde{Q} = 0$, which is a much weaker requirement).

The preceding observation has far-reaching consequences. It not only proves the Poisson-commutativity of the conserved charges Q given in (42) (–hence the

integrability of (37); the freedom of choosing $Q(x)$ in \mathbb{R}^3 e.g. as $\sum_{l=0, |m| \leq l}^{\infty} q_{lm} Y_{lm}(\vec{x})$ with $Y_{lm}(\vec{x})$ being the solid spherical harmonics, neatly matches the degrees of freedom to be expected from the one single field that is left over after the Hamiltonian reduction—) but —confirming that integrable theories come in hierarchies— shows that *any* of the charges (42) with $\Delta Q = 0$ may be considered as the Hamiltonian of another diffeomorphism invariant field theory (possessing (42) as commuting conserved charges).

Due to

$$\int_{\Sigma} d^M \varphi Q(x) (4g[x]p^2)^{\frac{1}{4}} = \int_{\Sigma} d^M \varphi (4\tilde{g}[x]\tilde{p}^2)^{\frac{1}{4}} =: \tilde{H}[x, p] \quad (48)$$

corresponding to $\tilde{Q}(x) = 1$ for a Riemannian manifold $(\mathcal{N}, \tilde{\zeta})$ with conformally equivalent metric

$$\tilde{\zeta}_{ij}(x) := (Q(x))^{\frac{4}{N-2}} \zeta_{ij}(x) \quad (49)$$

(implying $\sqrt{\tilde{g}} = (Q(x))^{2\frac{N-1}{N-2}} \sqrt{g}$, $\tilde{p}^2 = \tilde{p}_i \tilde{p}_j \tilde{\zeta}^{ij} = p_i p_j \zeta^{ij} Q^{-\frac{4}{N-2}}$; $N \neq 2$; the very special case $N = 2$ can be dealt with separately), and the fact that under the conformal change (49) the scalar curvature changes according to

$$\tilde{R} = Q^{-\frac{4}{N-2}} \left(R - 4 \frac{N-1}{N-2} \frac{\Delta Q}{Q} \right) \quad (50)$$

$$= Q^{-\frac{4}{N-2}} R \quad (51)$$

one sees that each hierarchy (of integrable Hamiltonian systems, resp. harmonic time-functions) consists of hypersurface motions in Riemannian manifolds having conformally equivalent metrics and satisfying

$$\tilde{R} \tilde{\zeta}_{ij} = R \zeta_{ij}. \quad (52)$$

This intriguing observation should have many important consequences. Taking e.g. $\mathcal{N} = \mathbb{R}^3$ and $\zeta_{ij} = \delta_{ij}$ the above suggests that the solutions of the linear, but nontrivial, equation

$$\vec{\nabla} \cdot (Q^2(\vec{x}) \vec{\nabla} \tau(\vec{x})) = 0 \quad (53)$$

resp. the corresponding nonlinear equations

$$\frac{\partial \vec{x}}{\partial t} = Q^2(\vec{x}) \frac{\sqrt{\det(\partial_r \vec{x} \cdot \partial_s \vec{x})}}{\rho} \vec{n} \quad (54)$$

are related to

$$\vec{\nabla}^2 \tau(\vec{x}) = 0 \quad (55)$$

resp.

$$\frac{\partial \vec{x}}{\partial t} = \frac{\sqrt{\det(\partial_r \vec{x} \cdot \partial_s \vec{x})}}{\rho} \vec{n} \quad (56)$$

via some generalized Bäcklund transformation for all $Q(\vec{x})$ satisfying

$$\vec{\nabla}^2 Q(\vec{x}) = 0. \quad (57)$$

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